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# Time-independent invariants of motion for the quadratic system

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**Abstract.** A Hamiltonian method is developed to obtain first integrals of the form $PQ^{\mu}$ and $PQ^{\mu}R^{\nu}$ for a general system of two-dimensional autonomous ordinary differential equations with quadratic terms, where $P$, $Q$ and $R$ are linear or quadratic polynomials, and $\mu$, $\nu$ are two real numbers. It is found that there is an intimate relationship between the polynomials and the equilibrium points of the system, which is useful for determining the existence of periodic orbits and asymptotic behaviour.

## 1. Introduction

In this work, we try to find invariants of the quadratic system (QS) defined by

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &\equiv \dot{x} = a_1 x + b_{11}x^2 + b_{12}xy + c_1 y^2 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &\equiv \dot{y} = a_2 y + c_2 x^2 + b_{21}xy + b_{22}y^2. \end{aligned} \tag{1}$$

There are many natural phenomena that can be described by the QS or equivalent systems [1]. For example, the Lotka–Volterra system (LVS) (with $c_1 = c_2 = 0$ in the QS) could model the time evolutions of conflicting species in biology and of chemical reactions [2]; the QS is derivable from the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics (with the assumption of quasi-neutrality to eliminate either the ion or electron equation) [3, 4], and a reduced QS is obtained from a generalized Blasius equation for fluid flow around a wedge-shaped obstacle in boundary layer theory [5]. In the context of conflicting species, the linear terms denote the growth (or decay) rates of each species independent of the other species; the self-interaction terms ($x^2$ in $\dot{x}$ and $y^2$ in $\dot{y}$) represent the control on over-population of each of the respective species (such as cannibalism or depletion of resources), and the cross-interaction terms ($xy$ in $\dot{x}$ and $\dot{y}$) represent inter-species (e.g predator–prey) interactions. In the context of plasma physics, all nonlinear terms represent binary interactions or model transport across the boundary of the system. In solving for the solutions of a QS, it is worth knowing, given a set of initial conditions, what its long-time asymptotic behaviour will be or whether stable periodic solutions exist. The existence of stable periodic orbits would be rather important for experimentalists wishing to obtain and maintain a stable oscillatory state. Since the solutions of the QS in general cannot be written in terms of elementary functions, the two questions of asymptotic behaviour and the existence of stable periodic orbits are rather hard to answer. Nor is it easy to explore the general solution using numerical schemes, since one

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has to prescribe all the coefficients of the QS in terms of real numbers. On the other hand, it is known that given certain conditions on the coefficients of the QS, one can obtain time-independent first integrals or invariants in terms of elementary functions ($I(x, y)$ such that $\dot{I} = 0$) [6]. Then trajectories of the QS are obtained as contours of the invariant, and, given a set of initial conditions, one can readily establish the existence of periodic solutions or find the asymptotic state. There is a long history of research on finding sufficient conditions for which periodic solutions (i.e. centres) exist for systems equivalent to the QS, and numerous results were obtained which we are not able to survey in full. However, most of the previous work assumed that the origin is a linear centre (i.e. having eigenvalues $\pm$i) which we do not assume here in equation (1). Reyn has compiled an excellent bibliography of such publications [7]. We mention two cases well known to us: one is the Carleman method [8] which assumes a quasi-polynomial form with unknown parameters for the invariant, and substitute it into the QS to solve for the unknowns and to obtain conditions on the QS coefficients. The Carleman method has been used successfully to find invariants (both time dependent and independent) in many dynamical systems, for example, the Lorenz system [9] and the $N$-dimensional LVS [2]. Recently, Cairó *et al* and Bouquet and Dewisme have developed a Hamiltonian method to find time-independent invariants [6, 10, 11]. Although the equations to solve for the unknown parameters for the Hamiltonian method are nonlinear (but algebraic) and hence more difficult than those of the Carleman method, which are linear, the advantage of the Hamiltonian system is that it can assume more general forms for the invariants than the Carleman method (see discussions in [6] and section 5 in this paper). Also in [6], Cairó *et al* discovered a geometrical relationship between time-independent invariants and the equilibrium points of the system (i.e. $\dot{x}(x_0, y_0) = \dot{y}(x_0, y_0) = 0$). Specifically, if an invariant of the form $I_1 = P_1(x, y)Q_1(x, y)^\mu$ exists (subject to certain conditions on the coefficients of the QS), for a quadratic polynomial $P_1(x, y)$ and a linear polynomial $Q_1(x, y)$ in $(x, y)$ and a number $\mu$, then $P_1(x, y) = 0$ is a conic section connecting two equilibrium points of the QS, and $Q_1(x, y) = 0$ is a line connecting the same two equilibrium points. In addition, if an invariant of the form $I_2 = P_2(x, y)Q_2(x, y)^\mu R_2(x, y)^\nu$ exists (subject to other conditions on the coefficients of the QS) for linear polynomials $P_2(x, y)$, $Q_2(x, y)$ and $R_2(x, y)$ in $(x, y)$, $\mu$ and $\nu$ being two numbers, then $P_2(x, y) = 0$, $Q_2(x, y) = 0$ and $R_2(x, y) = 0$ are three non-parallel lines joining three equilibrium points of the QS, thus forming a triangle. Since the polynomial curves $P_i(x, y) = 0$, $Q_i(x, y) = 0$, and $R_i(x, y) = 0$ ($i = 1$ or 2) of invariants $I_1(x, y)$ and $I_2(x, y)$ are contours of $I_1(x, y)$ (or $I_2(x, y)$), they are trajectories of the QS. For the invariants $I_2$, since the three lines form a triangle in phase space with vertices at three equilibrium points of the QS, the trajectories starting inside the triangle will evolve asymptotically to an attracting equilibrium point (e.g a sink or a stable node) at one vertex of the triangle. For invariants $I_1$, when the conic $P_1(x, y) = 0$ and the line $Q_1(x, y) = 0$ bound a region in phase space and $\mu > 0$, there exist stable periodic solutions. Otherwise if $\mu < 0$, trajectories starting inside the enclosed region will asymptotically approach an attracting equilibrium point, also at an intersection of $P_1 = 0$ and $Q_1 = 0$. We use these geometrical ideas to find more invariants of the forms $I_1$ and $I_2$ for the QS than those previously found in [6].

Here we consider polynomial curves that do intersect the origin ($S_i(0, 0) = 0$) in searching for invariants of the form $I_1$ and $I_2$, since in [6] we considered only polynomial curves $S_i(x, y) = 0$ (where $I \equiv S_1(x, y)S_2(x, y)^\mu \cdots$) that do not pass through the origin (i.e. $S_i(0, 0) \neq 0$). The organization of the paper is as follows: in section 2 we review the Hamiltonian method for the two invariants $I_1(x, y)$ and $I_2(x, y)$ of [6], discuss their geometrical properties via two sample phase portraits, and offer new interpretation of previous results pertaining to the existence of periodic orbits. In section 3, using the

Hamiltonian method, we calculate invariants of the form $I_1(x, y)$ with $P_1(0, 0) = 0$ and $Q_1(0, 0) = 0$ and the invariant conditions under which $I_1$ exists, and present plots and phase portraits showing that the intersections of the curves $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ are two equilibrium points of the QS, one of which is the origin. In section 4, we similarly calculate invariants of the form $I_2(x, y)$ with $P_2(0, 0) \neq 0$, $Q_2(0, 0) = 0$ and $R_2(x, y) = 0$ and the invariant conditions under which $I_2$ exists, and present a plot and phase portrait showing that the intersections of the curves $P_2(x, y) = 0$, $Q_2(x, y) = 0$ and $R_2(x, y) = 0$ are at three equilibrium points of the QS, forming a triangle with the origin at one vertex. In section 5, specializing to the LVS (a special case of the QS with $c_1 = c_2 = 0$), we apply the Hamiltonian method to search for invariants $I_1(x, y)$ and $I_2(x, y)$ and compare with those found previously in [2]. In section 6, we summarize the results, discuss the implications of the geometric properties on the existence of other invariants, and outline perspectives of future work.

## 2. Review of previous results of the Hamiltonian method for the QS

Given the QS expressed by equations (1), the Hamiltonian method [10] attempts to express a QS in the form

$$\begin{aligned} \dot{x} &= \frac{\partial I(x, y)}{\partial y} F(x, y) \\ \dot{y} &= -\frac{\partial I(x, y)}{\partial x} F(x, y) \end{aligned} \tag{2}$$

where $I(x, y)$ is a time-independent invariant and $F(x, y)$ is an arbitrary function of $(x, y)$ only. In [6], we tried to find an invariant of the form $I_1(x, y) = P_1(x, y)\, Q_1(x, y)^\mu$ where

$$\begin{aligned} P_1(x, y) &= k + Ax + By + Cx^2 + Dxy + Ey^2 \\ Q_1(x, y) &= 1 + \alpha x + \beta y \end{aligned} \tag{3}$$

are polynomials in $(x, y)$. Substituting $I_1(x, y)$ into equations (2), we arrived at

$$\begin{aligned} \dot{x} &= \left( \frac{\partial P_1}{\partial y} Q_1 + \mu \frac{\partial Q_1}{\partial y} P_1 \right) Q_1^{\mu-1} F \\ \dot{y} &= -\left( \frac{\partial P_1}{\partial x} Q_1 + \mu \frac{\partial Q_1}{\partial x} P_1 \right) Q_1^{\mu-1} F \end{aligned} \tag{4}$$

and substituting $P_1(x, y)$ and $Q_1(x, y)$ from (3) into (4), we noted that $Q_1^{\mu-1} F$ must be a constant for (4) to be equivalent to the QS. We took $Q_1^{\mu-1} F = 1$. Furthermore, equating the coefficients of $x^i y^j$ $(i, j = 0$ to $2)$ in (4) to the corresponding coefficients of $x^i y^j$ in (1) and solving, we arrived at

$$\mu = \frac{b_{11}^2 + b_{11}b_{21} + c_2 b_{22}}{b_{22}c_2 - b_{11}^2} \qquad k = \frac{-2c_2}{\mu(\mu+1)(\mu+2)\alpha^3}$$

$$A = -k\mu\alpha \qquad B = -k\mu\beta \qquad C = \frac{k}{2}\mu(\mu+1)\alpha^2$$

$$D = k\mu(\mu+1)\alpha\beta + a_1 \qquad E = \frac{k}{2}\mu(\mu+1)\beta^2$$

$$\alpha = -\frac{b_{21}b_{11} + 2b_{22}c_2}{a_1 b_{11}(\mu+1)} \qquad \beta = -\frac{\alpha b_{22}}{b_{11}}$$

and the invariant conditions,

$$a_1 + a_2 = 0 \qquad \frac{b_{21}}{b_{12}} = \frac{b_{11}}{b_{22}} = \left(\frac{c_2}{c_1}\right)^{1/3}.$$

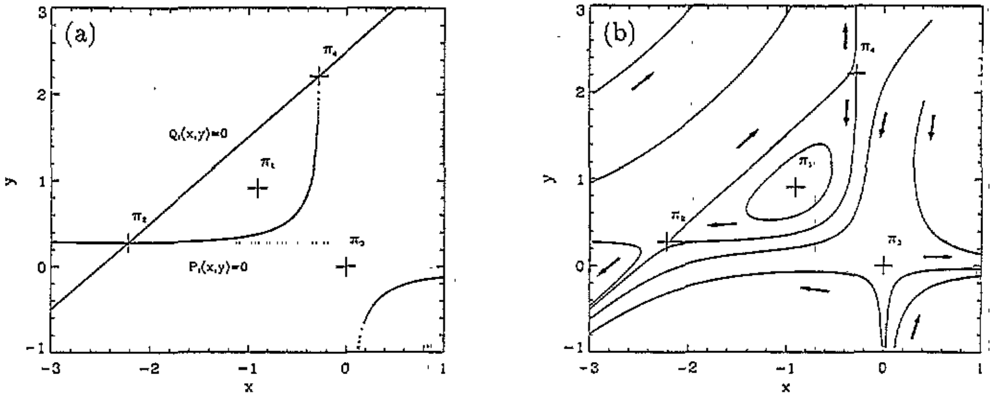


**Figure 1.** (*a*) Plot of $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ for $I_1 = P_1 Q_1^{\mu}$. (*b*) Phase portrait for the QS in the neighbourhood of the equilibrium points. Both plots use coefficients defined in section 2.

Also in [6] we showed that the conic section $P_1(x, y) = 0$ and the line $Q_1(x, y) = 0$ intersect at the same two equilibrium points of the QS. To show a sample plot of the conic and line, and the phase portrait of the QS, we use QS coefficients $a_1 = 1$, $b_{11} = 0.35$, $b_{22} = 0.35$, $b_{12} = -0.8$, $b_{21} = -0.8$, and hence the invariant conditions demand that $a_2 = -1$, $c_1 = -0.05$ and $c_2 = -0.05$. With these coefficients of the QS, $\mu = 1.25$ and there are four real equilibrium points at centre $\pi_1 = (-0.91, 0.91)$, saddle $\pi_2 = (-2.2, 0.28)$, saddle $\pi_3 = (0, 0)$ and saddle $\pi_4 = (-0.28, 2.2)$. Figure 1(*a*) shows the conic $P_1(x, y) = 0$ (in this case a hyperbola) and the line $Q_1(x, y) = 0$ which intersect at two equilibrium points $\pi_2$ and $\pi_4$; the conic and the line form the boundary of a closed region around the centre $\pi_1$. The conic $P_1(x, y) = 0$ and the line $Q_1(x, y) = 0$ are trajectories of the QS since they are contours of $I_1 = P_1(x, y)Q_1(x, y)^{\mu} = 0$ (since $\mu > 0$), hence trajectories starting inside the enclosed region (corresponding to different contours of $I_1$) cannot intersect either the conic or the line. Figure 1(*b*) shows the phase portrait of the QS in the neighbourhood of the equilibrium points. We note that since the centre $\pi_1$ is enclosed by the conic and the line, then there are stable periodic trajectories around $\pi_1$. It must be pointed out, however, that from our experience in plotting phase portraits, it seems unnecessary for $P_1 = 0$ and $Q_1 = 0$ to enclose a region of phase space in order to have stable periodic orbits (see figure 1 in [6]). Without making the assumption that $P_1 = 0$ and $Q_1 = 0$ enclose a region, we cannot provide any geometrical argument for the existence of stable periodic orbits for $\mu > 0$ (nor could we easily present any algebraic argument). Here we have implicitly assumed that $P_1(x, y) = 0$ does not intersect the origin, hence $P_1(0, 0) = k \neq 0$. Note that if $P_1(0, 0) = k = 0$, then $A = 0 = B = C = E$, and the invariant conditions also demand that $c_1 = 0 = c_2$, which would reduce the QS and the invariant $I_1(x, y)$ to the LVS and an invariant of type (III), respectively, as discussed in [2]. To obtain new results, the conic $P_1(x, y) = 0$ must not intersect the origin (i.e. $P_1(0, 0) = k \neq 0$). The line $Q_1(x, y)$ by construction does not intersect the origin.

In [6], we also tried to find an invariant of the form $I_2 = P_2(x,y)Q_2(x,y)^{\mu}R_2(x,y)^{\nu}$, where

$$\begin{aligned} P_2(x,y) &= 1 + \alpha x + \beta y \\ Q_2(x,y) &= 1 + Ax + By \\ R_2(x,y) &= 1 + Cx + Dy \end{aligned} \tag{5}$$

are linear polynomials in $(x, y)$. Similarly, we substituted $I_2(x, y)$ into (2) and arrived at

$$\begin{aligned} \dot{x} &= \left(\frac{\partial P_2}{\partial y}Q_2R_2 + \mu P_2R_2\frac{\partial Q_2}{\partial y} + \nu P_2Q_2\frac{\partial R_2}{\partial y}\right)Q_2^{\mu-1}R_2^{\nu-1}F \\ \dot{y} &= -\left(\frac{\partial P_2}{\partial x}Q_2R_2 + \mu P_2R_2\frac{\partial Q_2}{\partial x} + \nu P_2Q_2\frac{\partial R_2}{\partial x}\right)Q_2^{\mu-1}R_2^{\nu-1}F. \end{aligned} \tag{6}$$

In order that (6) is equivalent to a QS, we use reasoning similar to that above and set $Q_2^{\mu-1}R_2^{\nu-1}F = \text{constant} = 1$. Equating the coefficients of $x^iy^j$ in (6) to the corresponding coefficients of $x^iy^j$ in (1) and solving, we unfortunately could only obtain an implicit relation between the coefficients of $P_2(x, y)$, $Q_2(x, y)$ and $R_2(x, y)$ and those of the QS (for more details, see [6]). However, we did obtain explicitly the invariant conditions,

$$a_1 + a_2 = 0 \qquad b_{12} = b_{21} = 0.$$

We also showed that the lines $P_2(x, y) = 0$, $Q_2(x, y) = 0$ and $R_2(x, y) = 0$ intersect at the three equilibrium points of the QS excluding the origin, thus forming a triangle. Furthermore, we discovered that either $\mu$ or $\nu$ is negative (or both) and hence not all three lines are contours of $I_2 = P_2\,Q_2^{\mu}\,R_2^{\nu} = 0$ (the line with the negative exponent is a contour of $I_2 = \infty$) and thus trajectories starting inside the triangle will evolve to the attractive equilibrium point at a vertex of the triangle, even though they still cannot intersect the lines. To show a sample plot of the lines and phase portrait of the QS, we use $a_1 = 1.634$, $b_{11} = 2.80$, $b_{22} = 3.36$ $c_1 = 0.564$ and $c_2 = 0.376$ and the invariant conditions lead to

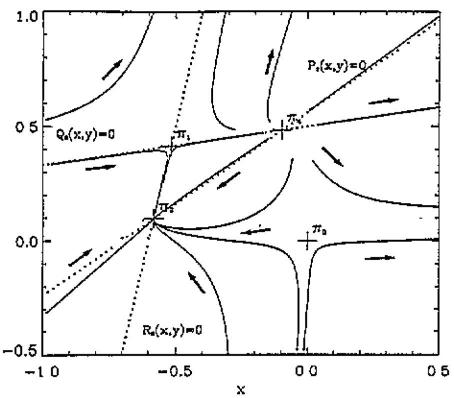


**Figure 2.** Plot of $P_2(x, y) = 0$, $Q_2(x, y) = 0$ and $R_2(x, y) = 0$ (dotted lines) and phase portrait for the QS in the neighbourhood of the equilibrium points. The coefficients are defined in section 2.

$a_2 = -1.634$ (along with $b_{12} = 0 = b_{21}$). The equilibrium points of the QS for this choice of coefficients are a saddle $\pi_1 = (-0.52, 0.41)$, a stable node $\pi_2 = (-0.58, 0.097)$, a saddle $\pi_3 = (0, 0)$, and an unstable node $\pi_4 = (-0.097, 0.48)$. With these coefficients of the QS, the resulting solutions give the lines $P_2(x, y) = 1 + 1.43x - 1.78y = 0$, $Q_2(x, y) = 1 + 0.33x - 2y = 0$ and $R_2(x, y) = 1 + 1.67x - 0.33y$ intersecting at $\pi_1$, $\pi_2$ and $\pi_4$, forming a triangle as shown in figure 2 (dotted lines). The phase portrait is also shown in figure 2, where trajectories forming the triangle with vertices at $\pi_1$, $\pi_2$ and $\pi_4$ are evident. Also, all trajectories starting inside the triangle approach the vertex $\pi_2$ (the stable node). Once again, the lines do not intersect the origin, since we had chosen $P_2(0, 0)$, $Q_2(0, 0)$ and $R_2(0, 0) \neq 0$.

## 3. Hamiltonian method for more invariants of the form $I_1 = P_1(x, y)Q_1(x, y)^\mu$

Since in [6] we imposed that in (3) $P_1(0, 0) \neq 0$ and $Q_1(0, 0) \neq 0$, the curves $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ cannot intersect at the origin. However, the geometry of the phase portrait in figure 1($a$) seems to suggest that there are other invariants of the form $I_1(x, y)$ whose polynomial curves $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ do intersect the origin (i.e. $P_1(0, 0) = 0$ and $Q_1(0, 0) = 0$). In fact, based on the geometrical assumption that the origin is one of two equilibrium points which are the intersections of a conic and a line, we expect to find three such cases in general (since the QS has in general four equilibrium points). Thus we now try

$$\begin{aligned} &P_1(x, y) = Ax + By + Cx^2 + Dxy + Ey^2 \\ &Q_1(x, y) = \alpha x + \beta y. \end{aligned} \tag{7}$$

Note that both factors are zero at the origin. Substituting $P_1(x, y)$ and $Q_1(x, y)$ into (4) (setting $Q_1^{\mu-1}F = 1$, as before) and equating with the corresponding equations in (1), we get

$$\begin{aligned} &\mu\beta A + \alpha B = a_1 \qquad \beta B(\mu + 1) = 0 \\ &\alpha D + \mu\beta C = b_{11} \qquad \mu\beta E + 2E\beta = c_1 \\ &2\alpha E + \beta D + \mu\beta D = b_{12} \end{aligned} \tag{8$a$}$$

$$\begin{aligned} &\alpha A(\mu + 1) = 0 \qquad \beta A + \alpha\mu B = -a_2 \\ &\mu\alpha C + 2\alpha C = -c_2 \qquad \mu\alpha E + \beta D = -b_{22} \\ &\mu\alpha D + 2\beta C + \alpha D = -b_{21} \end{aligned} \tag{8$b$}$$

where (8$a$) represent the coefficients of $x$, $y$, $x^2$, $y^2$ and $xy$, respectively, in the equation for $\dot{x}$ and (8$b$) represent the coefficients of $x$, $y$, $x^2$, $y^2$ and $xy$, respectively, in the equation for $\dot{y}$. We first examine a set of two homogeneous equations

$$\beta B(\mu + 1) = 0 \qquad \alpha A(\mu + 1) = 0$$

which has five possible solutions $[\alpha = 0, \beta = 0]$, $[\alpha = 0, B = 0]$, $[A = 0, \beta = 0]$, $[\mu = -1]$ and $[A = 0, B = 0]$, where the first is trivial since it leads to $Q_1(x, y) \equiv 0$ and will not be considered. The next three cases are the three different geometrical configurations where the origin is one of the two equilibrium points intersected by $P_1(x, y) = 0$ and

$Q_1(x, y) = 0$. The last case is degenerate and will be discussed in detail. We consider the four cases separately.

(i) $\alpha = 0$, $B = 0$

This case leads readily to the solutions of (8*a*) and (8*b*):

$$A = E\frac{a_1 - 2a_2}{c_1} \qquad C = E\frac{b_{11}(a_1 - 2a_2)}{a_1 c_1}$$
$$D = E\frac{b_{22}(a_1 - 2a_2)}{a_2 c_1} \qquad E \sim \text{arbitrary}$$
$$\beta = -\frac{a_2 c_1}{E(a_1 - 2a_2)} \qquad \mu = -\frac{a_1}{a_2}$$

and invariant conditions

$$b_{12} = \frac{(a_1 - a_2)b_{22}}{a_2} \qquad b_{21} = \frac{2b_{11}a_2}{a_1} \qquad c_2 = 0.$$

Because of the order in which we solved (8), all the coefficients of $P_1(x, y)$ are proportional here to an arbitrary constant $E$, which is the coefficient of $y^2$ in $P_1(x, y)$. However, if we solve (8) in a different order, another coefficient of $P_1$ (rather than $E$) would be arbitrary, and the rest of the coefficients would be proportional to it. In fact, this is not surprising since whenever a coefficient of $P_1$ (or $Q_1$) is non-zero it can always be factored out and ignored, and the invariant $I_1$ would still remain an invariant. What is important is the relationship between the coefficients, which remains the same for any non-zero value of the arbitrary constant. Furthermore, since $\alpha = 0$ in $Q_1(x, y)$, $\beta$ could also be factored from $Q_1$. Hence the solutions to the coefficients of $P_1(x, y)$ and $Q_1(x, y)$ could be written alternatively as (with $E$ and $\beta$ factored out)

$$A = \frac{a_1 - 2a_2}{c_1} \qquad B = 0$$
$$C = \frac{b_{11}(a_1 - 2a_2)}{a_1 c_1} \qquad D = \frac{b_{22}(a_1 - 2a_2)}{a_2 c_1} \qquad E = 1$$
$$\alpha = 0 \qquad \beta = 1 \qquad \mu = -\frac{a_1}{a_2}.$$

The proportionality of all coefficients in $P_1$ to an arbitrary constant and the subsequent factoring of the arbitrary constant imply that the number of unknown coefficients in $P_1$ is one less than originally assumed, from five ($A$, $B$, $C$, $D$ and $E$) to four ($A$, $B$ $C$ and $D$). That $\beta$ could be factored from $Q_1$ also reduces the number of unknowns in $Q_1$ from two ($\alpha$ and $\beta$) to one ($\alpha$). The factorizable property of a coefficient, arbitrary or not, from $P_1$ and $Q_1$ is present for the rest of the cases considered (also for $Q_2$ and $R_2$ in section 4). The actual number of unknown coefficients versus the number of equations might have important implications for the existence of invariants, a conjecture we will discuss in section 6. To show a sample plot, we choose the coefficients of the QS as $a_1 = 3$, $a_2 = 2$, $b_{11} = 1$, $b_{22} = 2$, $c_1 = -1$ and, hence, $b_{12} = 1$ and $b_{21} = \frac{4}{3}$ from the invariant conditions. With these coefficients, $\mu = -\frac{3}{2} < 0$ and the real equilibrium points of the QS and their linear analyses are an unstable node $\pi_1 = (0, 0)$, a stable node $\pi_2 = (-3, 0)$ and a double equilibrium saddle point at $\pi_3 = (3, -3)$. Figure 3(*a*) presents the phase portrait and shows the line $Q_1(x, y) = \beta y = 0$, the $x$-axis, and the conic section $P_1(x, y) = 0$, an ellipse with this choice of coefficients of the QS ($E = 1$). The intersections of $Q_1(x, y) = 0$ and $P_1(x, y) = 0$ are $\pi_1$ and $\pi_2$ and indeed are two equilibrium points of the QS. Here, similarly to figure 1($a$), the phase space bounded by $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ is a closed region. But, in contrast to figure 1($b$), where $\mu > 0$ and hence both $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ are contours of $I_1 = 0$, and trajectories starting inside the bounded region cannot intersect the boundary, here $\mu < 0$, hence $P_1(x, y) = 0$ is a contour of $I_1 = 0$ and $Q_1(x, y) = 0$ is a contour of $I_1 = \infty$, and trajectories starting inside the bounded region could evolve to the boundary at the stable node $\pi_2$ (the intersection of $P_1 = 0$ and $Q_1 = 0$). To illustrate the importance of the sign of $\mu$ on the existence of stable periodic trajectories, we present in figure 3($b$) a sample phase portrait with $a_1 = 2$, $a_2 = -1.7$, $b_{11} = -2.5$, $b_{22} = 1.4$, $c_1 = -1.6$ and, hence, $b_{12} = -3.1$ and $b_{21} = 4.1$ from the invariant conditions. Here $\mu = 1.2 > 0$; the ellipse $P_1(x, y) = 0$ and line $Q_1(x, y) = 0$, both trajectories and contours of $I_1 = 0$, intersect at two saddles $\pi_3$ and $\pi_4$ and enclose a region containing two centres $\pi_1$ and $\pi_2$. There are stable periodic orbits around the two enclosed centres. In contrast to figure 3($a$), where other equilibrium points are outside the enclosed region and trajectories starting inside evolve to the stable node, here all equilibria are inside and on the boundaries of the enclosed region and trajectories starting inside are closed and periodic. The feature that polynomial curves enclose a bounded region in the neighbourhood of equilibrium points is quite useful in that all trajectories starting inside the region either remain inside, where, if there is a centre (see figures 1($b$), 3($b$)), it would confirm the existence of stable periodic trajectories, or evolve to the attractive equilibrium point (an intersection of $P_1(x, y) = 0$ and $Q_1(x, y) = 0$) which would yield their asymptotic behaviour as seen in figure 3($a$). Some subsequent phase portraits for other cases (with appropriate coefficients of the QS) also exhibit this useful feature.

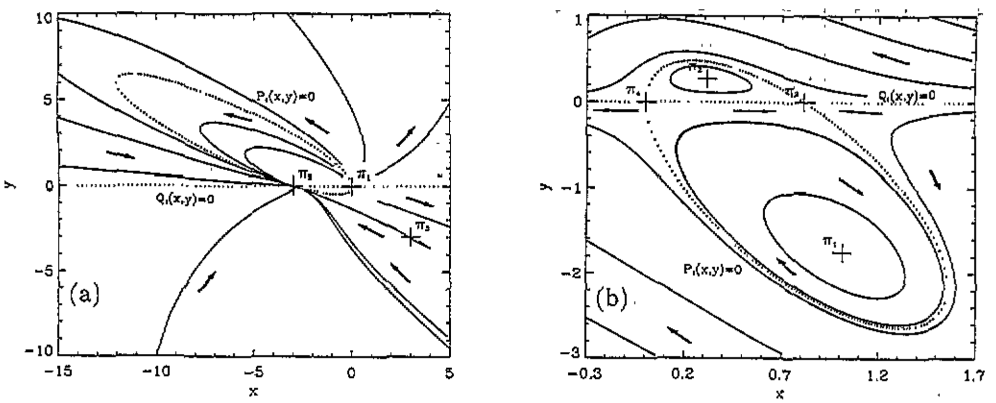


**Figure 3.** ($a$) Plot of $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ (dotted curves) and phase portrait for the QS about the neighbourhood of the equilibrium points. Coefficients are defined in section 3, case (i), with $\mu < 0$. ($b$) Plot of $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ (dotted curves) and phase portrait for the QS about the neighbourhood of the equilibrium points. Coefficients are defined in section 3, case (i), with $\mu > 0$.

(ii) $A = 0, \beta = 0$

This is the 'symmetric' case to (i) and has the following solutions to (8$a$) and (8$b$):

$$B = \frac{a_2 E}{b_{22}} \qquad C = -\frac{a_2 c_2 E}{b_{22}(2a_1 - a_2)}$$

$$D = \frac{a_2 b_{11} E}{a_1 b_{22}} \qquad E \sim \text{arbitrary}$$
$$\alpha = \frac{a_1 b_{22}}{a_2 E} \qquad \mu = -\frac{a_2}{a_1}$$

and invariant conditions

$$b_{12} = 2\frac{a_1 b_{22}}{a_2} \qquad b_{21} = \frac{b_{11}(a_2 - a_1)}{a_1} \qquad c_1 = 0.$$

Because this is the symmetric case to (i), the arbitrary constant $E$ plays the same role in $P_1$ and could be set equal to 1, and $\alpha$, similar to $\beta$ in (i), could also be set equal to 1. To show different characteristics, the sample plot here uses $a_1 = 3$, $a_2 = -2.41$, $b_{11} = 0.679$, $b_{22} = 2.23$, $c_2 = -1.67$ and thus, $b_{12} = -5.54$ and $b_{21} = -1.23$ from the invariant conditions. With these coefficients $\mu > 0$, the real equilibrium points of the QS and their linear analyses are a saddle $\pi_1 = (0, 0.83)$ and $\pi_2 = (0, 0)$, also a saddle. Figure 4 shows the phase portrait and the line $Q_1(x, y) = 0$ and conic section $P_1(x, y) = 0$ (both dotted curves) (also with $E = 1$), where the line $Q_1(x, y) = \alpha x = 0$ is the $y$ axis and the conic section $P_1(x, y) = 0$ is a hyperbola (for these chosen coefficients of the QS). The intersections of $Q_1(x, y) = 0$ and $P_1(x, y) = 0$ are $\pi_1$ and $\pi_2$, the two equilibrium points of the QS. Unlike figure 3, here the polynomial curves do not enclose a bounded region in phase space.

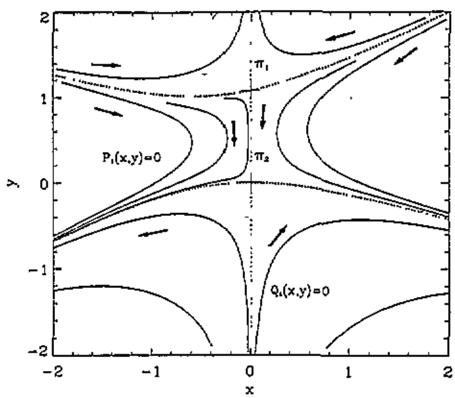


**Figure 4.** Plot of $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ (dotted curves) and phase portrait for the QS in the neighbourhood of the equilibrium points. Coefficients are defined in section 3, case (ii).

(iii) $\mu = -1$

This leads to the following solutions of (8*a*) and (8*b*):

$$A \sim \text{arbitrary} \qquad B = \frac{2(a_1 E + c_1 A)}{b_{12}}$$
$$C = -\frac{b_{21} E}{2c_1} \qquad D = \frac{(2b_{11} - b_{21})E}{b_{12}} \qquad E \sim \text{arbitrary}$$
$$\alpha = \frac{b_{12}}{2E} \qquad \beta = \frac{c_1}{E}$$

and the invariant conditions

$$a_1 = a_2 \qquad c_1 = -\frac{b_{12}(2b_{22} - b_{12})}{2(2b_{11} - b_{21})} \qquad c_2 = -\frac{b_{21}(2b_{11} - b_{21})}{2(2b_{22} - b_{12})}.$$

Here the arbitrary constant $E$ is factorizable from $P_1$, as in cases (i) and (ii), but $A$ is truly arbitrary. Also, either $\alpha$ or $\beta$ is factorizable from $Q_1$ (since $\alpha$ and $\beta$ cannot both be zero). Factoring $E$ and either $\alpha$ or $\beta$ from $P_1$ and $Q_1$, respectively, would give alternative expressions for the solutions to the coefficients as in case (i), which is equivalent to setting the arbitrary constants equal to 1. For a sample plot we use $a_1 = 3$, $b_{11} = 1$, $b_{12} = 1$, $b_{21} = 1$, $b_{22} = -0.5$ and hence $c_1 = 1$ and $c_2 = 0.25$ from the invariant conditions. With these coefficients, the real equilibrium points of the QS and their linear analyses are a saddle $\pi_1 = (-2, 2.73)$, a stable node $\pi_2 = (-4, 2)$, an unstable node $\pi_3 = (0, 0)$ and a saddle $\pi_4 = (-2, -0.73)$. Figure 5($a$) shows the intersections of the line $Q_1(x, y) = 0$ and a family of conic sections $P_1(x, y; A) = 0$ (with $E = 1$) at $\pi_2$ and $\pi_3$, two equilibrium points of the QS. Here the region asymptotically bounded by the family of polynomial curves is the parallelogram with vertices at the equilibrium points, and since $\mu < 0$, all trajectories starting inside evolve toward the stable node $\pi_2$. Using the Runge–Kutta method, figure 5($b$) shows the phase portrait of the QS which is nicely matched by the family of conics $P_1(x, y) = 0$ and the line $Q_1(x, y) = 0$ in figure 5($a$).

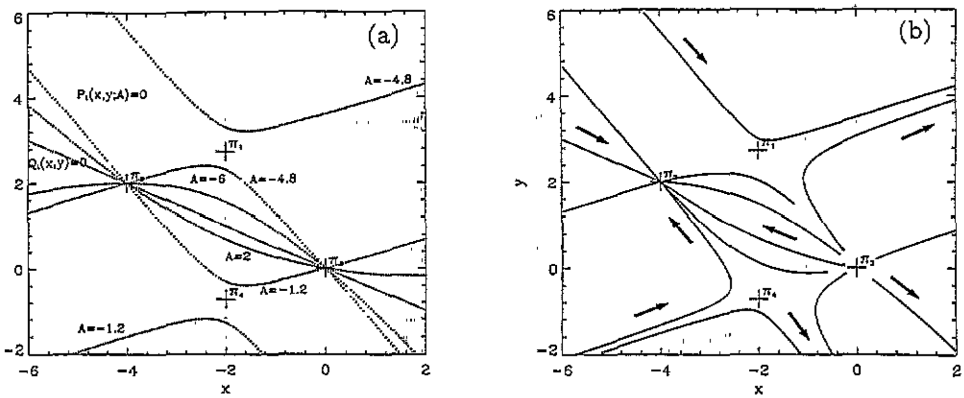


**Figure 5.** ($a$) Plot of the family of conics $P_1(x, y; A) = 0$ and $Q_1(x, y) = 0$; ($b$) phase portrait for the QS in the neighbourhood of the equilibrium points. Both plots use coefficients defined in case (iii), section 3.

(iv) $A = 0$, $B = 0$

This is a degenerate case since the quadratic polynomial $P_1(x, y)$ is factorizable into two homogeneous linear polynomials. This solution to the homogeneous equations also leads to a set of coupled nonlinear algebraic equations that have no simple expression for their solutions. Thus we simplify as much as possible the solutions to (8$a$) and (8$b$), which lead to

$$C = -\frac{c_2(1-\mu)E}{(1+\mu)b_{22} + b_{12}} \qquad D = -\frac{(2b_{22} + \mu b_{12})E}{c_1(1-\mu)} \qquad E \sim \text{arbitrary}$$
$$\alpha = \frac{(1+\mu)b_{22} + b_{12}}{(2+\mu)(1-\mu)E} \qquad \beta = \frac{c_1}{(2+\mu)E}$$
$$\mu = \frac{(2b_{11} + b_{21})b_{22} + b_{11}b_{12} + 2c_1c_2 \pm d}{2(c_1c_2 - b_{11}b_{22})}$$

where

$$d = [(b_{22}b_{21} - b_{11}b_{12})^2 + 4(2b_{22} + b_{12})(2b_{11} + b_{21})c_1c_2]^{1/2}$$

and invariant conditions

$$a_1 = 0 \qquad a_2 = 0$$

and

$$b_{11} = \frac{[b_{22}(1+\mu) + b_{12}]\,(2b_{22} + \mu b_{12})}{(2+\mu)(1-\mu)^2 c_1} - \frac{\mu(\mu-1)c_1c_2}{(2+\mu)[(1+\mu)b_{22} + b_{12}]}$$

where the third invariant condition is a complicated nonlinear equation in $b_{11}$ (since $\mu$, which appears on the RHS, is a function of $b_{11}$). As in previous cases (i) through (iii), $E$ is factorizable from $P_1$ and either $\alpha$ or $\beta$ is factorizable from $Q_1$, which would yield alternative expressions for the solutions to the coefficients that are equivalent to setting the arbitrary constants equal to 1. With invariant conditions $a_1 = 0$ and $a_2 = 0$, the equilibrium points of the QS become rather degenerate in that (0,0) is the only real equilibrium point. Furthermore, linear analysis about (0,0) shows that both eigenvalues are zero, hence the origin is a peculiar equilibrium point which we are unable to classify. Vulpe and Sibirskii have extensively studied phase portraits of QS with $a_1 = a_2 = 0$ [12]. Furthermore, with $A = 0$, $B = 0$, the conic section $P_1(x, y) = 0$ factorizes into two (real or complex) lines. The sample plot here uses $b_{12} = 5.36$, $b_{21} = 1.41$, $b_{22} = -1.5$, $c_1 = -2.92$, $c_2 = 0.266$ and hence $b_{11} = 0.33$ from the third invariant condition. Figure 6 is the phase portrait and shows the line $Q_1(x, y) = 0$ and the two real lines of $P_1(x, y) = 0$ (with $E = 1$) whose intersection is at $(0, 0)$, i.e. the sole equilibrium point of the QS. The three lines divide the phase space into six regions in which the trajectories behave as if the origin is a saddle. The three lines, being trajectories, do not change their flow direction after intersecting the origin.

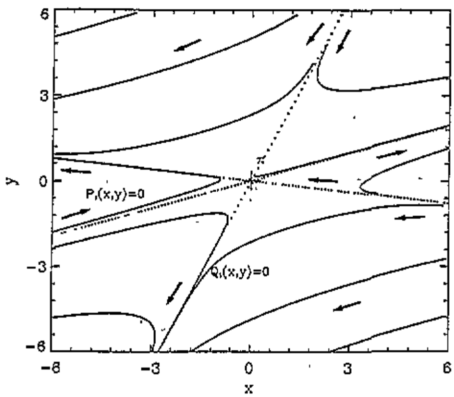


**Figure 6.** Plot of the two lines of $P_1(x, y) = 0$ and $Q_1(x, y) = 0$ (dotted lines) and phase portrait for the QS in the neighbourhood of the equilibrium point $(0, 0)$. Coefficients are defined in case (iv) of section 3.

In this section we have calculated all the new invariants of the form $I_1(x, y) = P_1(x, y)Q_1(x, y)^\mu$ in which $P_1(0, 0) = 0$ and $Q_1(0, 0) = 0$, where $P_1(x, y)$ is quadratic and $Q_1(x, y)$ is linear $(x, y)$. From geometric considerations, we expect, and find, the three cases for which the origin is one of two equilibrium points that are the intersections of the conic $P_1(x, y) = 0$ and the line $Q_1(x, y) = 0$, namely cases (i), (ii) and (iii). For those cases and for closed regions bounded by $P_1 = 0$ and $Q_1 = 0$, if $\mu > 0$ then stable periodic orbits exist inside. Case (iv) is a degenerate case of the invariant of the form $I_2$, which will be discussed further in section 4. To be complete, we now discuss two cases, $[P_1(0, 0) = 0,\ Q_1(0, 0) \neq 0]$ and $[P_1(0, 0) \neq 0$ and $Q_1(0, 0) = 0]$, and show that they both lead to inconsistencies. First, for the case $[P_1(0, 0) \neq 0$ and $Q_1(0, 0) = 0]$, we note that (after setting $Q_1^{\mu-1}F = 1$) (4) at the equilibrium point $(0, 0)$ reduce to

$$\mu P_1(0,0)\left[\frac{\partial Q_1}{\partial y}\right]_{(0,0)} = 0 \qquad \mu P_1(0,0)\left[\frac{\partial Q_1}{\partial x}\right]_{(0,0)} = 0.$$

Thus, either $\mu = 0$, which violates the functional form of $I_1(x, y)$, or $(\partial Q_1/\partial x) = 0$ and $(\partial Q_1/\partial y) = 0$ at $(0, 0)$, which leads to $Q_1(x, y) \equiv 0$ and hence to a trivial invariant, or, lastly, $P_1(0, 0) = 0$, which is a contradiction. The other case $[P_1(0, 0) = 0$ and $Q_1(0, 0) \neq 0]$ would reduce the QS to the LVS and has been discussed previously in section 2.

## 4. Hamiltonian method for more invariants of the form $I_2 = P_2(x, y)Q_2(x, y)^\mu R_2(x, y)^\nu$

Because we had imposed that $P_2(0, 0) \neq 0$, $Q_2(0, 0) \neq 0$ and $R_2(0, 0) \neq 0$ in (5), thus the lines $P_2(x, y) = 0$, $Q_2(x, y) = 0$ and $R_2(x, y) = 0$ could not intersect the origin. However, the geometry of the phase portrait in figure 2 seems to suggest that there are other invariants of the form $I_2(x, y)$ whose polynomial factors $P_2(x, y) = 0$, $Q_2(x, y) = 0$ and $R_2(x, y) = 0$ could intersect the origin. Based on the geometry that the origin is one of three equilibrium points forming a triangle, we expect to find three different configurations. We now try

$$P_2(x, y) = 1 + \alpha x + \beta y \qquad Q_2(x, y) = Ax + By \qquad R_2(x, y) = Cx + Dy. \tag{9}$$

Here, we choose only two lines, $Q_2(x, y) = 0$ and $R_2(x, y) = 0$, to intersect the origin, since the geometry of figure 2 suggested that each equilibrium point intersected by the lines is a vertex of a triangle. Substituting $P_2(x, y)$, $Q_2(x, y)$ and $R_2(x, y)$ (and setting $Q_2^{\mu-1}R_2^{\nu-1}F = 1$ as before) into (6) and equating with the corresponding equations in (1), we get

$$\begin{aligned}
&D\nu A + C\mu B = a_1 \qquad DB(\nu + \mu) = 0\\
&(C\beta + D\nu\alpha)A + C\mu B\alpha = b_{11} \qquad (\nu + 1 + \mu)D\beta B = c_1\\
&(\nu + \mu)D\alpha B + (1 + \nu)D\beta A + (1 + \mu)C\beta B = b_{12}
\end{aligned} \tag{10a}$$

$$\begin{aligned}
&\nu CB + DA\mu = -a_2 \qquad AC(\nu + \mu) = 0\\
&DA\mu\beta + (D\alpha + \nu C\beta)B = -b_{22} \qquad (\nu + 1 + \mu)C\alpha A = -c_2\\
&(\nu + \mu)C\beta A + (1 + \mu)D\alpha A + (1 + \nu)C\alpha B = -b_{21}
\end{aligned} \tag{10b}$$

where (10$a$) and (10$b$) represent the coefficients of $x$, $y$, $x^2$, $y^2$ and $xy$, respectively, in the $\dot{x}$ and $\dot{y}$ equations. As in section 3, we first examine the two homogeneous equations

$$BD(\mu+\nu)=0 \qquad AC(\mu+\nu)=0$$

which have five possible solutions, $[A=0, B=0]$, $[C=0, D=0]$, $[B=0, C=0]$, $[A=0, D=0]$ and $[\mu=-\nu]$, with the first two being trivial, since they lead to $Q_2(x,y)\equiv 0$ and $R_2(x,y)\equiv 0$, respectively, and thus will not be considered. Furthermore, the next two solutions lead to invariant conditions $c_1=0$, $c_2=0$, which transform the QS into the Lotka–Volterra equations, and $a_2b_{11}(b_{12}-b_{22})+a_1b_{22}(b_{21}-b_{11})=0$, which leads to invariants of type (III), previously obtained by Cairó and Feix in [2].

Hence we consider the last case $[\mu=-\nu]$. This solution leads to a set of coupled nonlinear algebraic equations from (10$a$) and (10$b$). As in case (iv) of section 3, we simplify as much as possible the expressions of the solutions to (10$a$) and (10$b$), which are

$$\begin{aligned}
\alpha &= \frac{-c_2b_{12}^2+(2b_{22}c_2+b_{11}b_{21})b_{12}-2c_1c_2b_{21}}{a_1(b_{12}b_{21}-4c_1c_2)}\\
\beta &= \frac{-c_1b_{21}^2+(2b_{11}c_1+b_{22}b_{12})b_{21}-2c_1c_2b_{12}}{a_2(b_{12}b_{21}-4c_1c_2)}\\
A &\sim \text{arbitrary} \qquad D\sim\text{arbitrary}\\
B &= \frac{c_1}{\beta D} \qquad C=-\frac{c_2}{\alpha A}\\
\mu &= -\frac{a_1}{(c_1c_2/\alpha\beta)+1}
\end{aligned}$$

and the invariant conditions

$$a_2=a_1$$

and

$$\frac{\beta}{b_{12}-\beta(2+a_1)}=-\frac{\alpha}{b_{21}+\alpha(2-a_1)}=-\frac{1}{(c_1c_2/\alpha\beta)+1}$$

where the second and third invariant conditions are complicated coupled nonlinear equations involving all the coefficients of the QS (with $\alpha$ and $\beta$ given above). As in previous cases in section 3, the arbitrary constants $A$ and $D$ are factorizable from $Q_2$ and $R_2$, respectively, or we may set $A$, $D=1$. Thus, the number of unknowns in $P_2$ is two, unaffected ($\alpha$ and $\beta$), whereas in $Q_2$, it is reduced from two ($A$ and $B$) to one ($B$) and in $R_2$ it is also reduced from two ($C$ and $D$) to one ($C$). Here the three lines $P_2(x,y)=0$, $Q_2(x,y)=0$ and $R_2(x,y)=0$ form a triangle as in the corresponding case in [6]. Also, since $\mu=-\nu$, only two of the lines ($P_2(x,y)=0$ and either $Q_2(x,y)=0$, if $\mu>0$, or $R_2(x,y)=0$, if $\nu>0$) are contours of $I_2=0$, with the third one being a contour of $I_2=\infty$. Hence, we expect trajectories starting inside the triangle to approach the attractive equilibrium point at one vertex of the triangle. Unlike similar cases in sections 2 and 3, where if $\mu>0$ periodic trajectories exist inside the bounded region, here there is no periodic trajectory inside the triangle since $\mu$ or $\nu$ is negative. For a sample plot we use $a_1=1$, $b_{11}=1$, $b_{12}=0.667$, $b_{21}=1$, and $b_{22}=-3.56$, hence $a_2=1$, $c_1=-7.40$ and $c_2=0.180$ from the invariant conditions. With these coefficients, the real equilibrium points of the QS and their linear analyses are a stable node $\pi_1 = (-1.56, 0.282)$, a saddle $\pi_2 = (-1.88, -0.563)$, a saddle $\pi_3 = (5.0, 2.25)$ and a source $\pi_4 = (0, 0)$. Figure 7 shows the phase portrait and the three lines $P_2(x, y) = 0$, $Q_2(x, y) = 0$ (with $A = 1$) and $R_2(x, y) = 0$ (with $D = 1$) forming a triangle. Their intersections at $\pi_1$, $\pi_3$ and $\pi_4$ are indeed three equilibrium points of the QS. Trajectories starting inside the triangle tend to the stable node $\pi_1$ asymptotically. From geometric considerations of the equilibrium points of the QS, we expect to find three different configurations of triangles with the origin at one vertex, but have presented only one. However, the solutions for the coefficients of the polynomials, along with the second and third invariant conditions involving $\alpha$, $\beta$, $b_{12}$ and $b_{21}$, could be transformed to a cubic equation in one of the variables (but too algebraically cumbersome to be presented) which implies three possible sets of (second and third) invariant conditions on the coefficients for the QS, representing the three configurations.

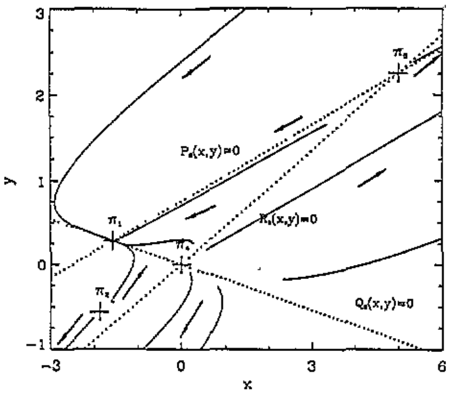


**Figure 7.** Plot of $P_2(x, y) = 0$, $Q_2(x, y) = 0$ and $R_2(x, y) = 0$ (dotted lines) and phase portrait for the QS in the neighbourhood of the equilibrium points. The coefficients are defined in section 4.

In this section we have calculated all new invariants of the form $I_2 = P_2(x, y)Q_2(x, y)^\mu R_2(x, y)^\nu$ in which $P_2(0, 0) \neq 0$, $Q_2(0, 0) = 0$ and $R_2(0, 0) = 0$, where $P_2(x, y)$, $Q_2(x, y)$ and $R_2(x, y)$ are up to linear orders in $(x, y)$. To complete the analysis, we now discuss two cases, where in one case, only one line intersects the origin, say $R_2(0, 0) = 0$ and $P_2(0, 0)$, $Q_2(0, 0) \neq 0$, and in the other case, all three lines intersect the origin. For the case $R_2(0, 0) = 0$ and $P_2(0, 0)$, $Q_2(0, 0) \neq 0$ (after setting $Q_2^{\mu-1} R_2^{\nu-1} F = 1$), equations (6) at the equilibrium point $(0, 0)$ reduce to

$$\nu P_2(0,0)Q_2(0,0)\left[\frac{\partial R_2}{\partial y}\right]_{(0,0)} = 0$$

$$\nu P_2(0,0)Q_2(0,0)\left[\frac{\partial R_2}{\partial x}\right]_{(0,0)} = 0.$$

Therefore, either $\nu = 0$, which violates the functional form of $I_2(x, y)$, or $(\partial R_2/\partial x) = 0$ and $(\partial R_2/\partial x) = 0$ at $(0, 0)$, which leads to $R_2(x, y) \equiv 0$ and hence a trivial invariant, or, lastly, $P_2(0, 0)Q_2(0, 0) = 0$, which is a contradiction. For the case where all three lines intersect the origin, the resulting solutions are a generalization of case (iv) in section 3, since there are now only two invariant conditions $a_1 = a_2 = 0$, instead of the three previous conditions with $a_1 = a_2 = 0$ and a complicated third invariant condition. We present the results for this degenerate case in the appendix.

## 5. Hamiltonian method for the Lotka–Volterra system

Here, we apply the Hamiltonian method to the Lotka–Volterra System (LVS) to find new invariants of the form $I_1(x, y)$ and $I_2(x, y)$. The LVS is a special case of the QS where $c_1 = c_2 \equiv 0$, and Cairó and Feix have found three types of invariants [2]. For simplicity, we apply the rescaling method [10] to the LVS, which reduces to

$$\begin{aligned}\frac{dX}{dT} &= X(1 + X + RY)\\ \frac{dY}{dT} &= AY(1 + R'X + Y)\end{aligned} \tag{11}$$

where $X = (b_{11}x/a_1)$, $Y = (b_{22}y/a_2)$, $T = (a_1 t)$, $R = (b_{12}a_2/b_{22}a_1)$ $R' = (b_{21}a_1/b_{11}a_2)$ and $A = (a_2/a_1)$. This rescaled LVS always has three equilibrium points at $(0, 0)$, $(-1, 0)$ and $(0, -1)$ and an equilibrium point at

$$\left(\frac{1-R}{R'R-1}, \frac{1-R'}{R'R-1}\right)$$

when $RR' \neq 1$. The equations to solve for the coefficients of invariants $I_1(x, y)$ and $I_2(x, y)$ are the same as (8), (10) and the corresponding equations in [6], with the coefficients of the LVS substituted appropriately, specifically $a_1 = 1$, $b_{11} = 1$, $b_{12} = R$, $c_1 = 0$, $a_2 = A$, $b_{22} = A$, $b_{21} = AR'$ and $c_2 = 0$. There are many invariants for this simplified system; however, many of the solutions are either trivial or special cases of those already found by Cairó and Feix [2] and therefore will not be presented here. For example, we discard cases for which one of the invariant conditions is $R = 0$ or $R' = 0$ (or both), because then one of the equations in (11) is immediately solvable and the other becomes a Riccati equation which has well known invariants. We also discard the case of $A = 1$, $R = 1 = R'$ which has the obvious invariant $X = CY$ ($C$ is a constant). There are only three new and non-trivial invariant conditions, all of them of the form $I_1 = P_1 Q_1^{\mu}$. They are as follows:

(i) Invariant conditions: $A = -1$, $R = -1$ and $R' = -1$,

$$I_1(X, Y) = \frac{(X+1)^2 + (Y+1)^2 - 1}{(1 + X + Y)^2}.$$

Here the circle $P_1(X, Y) = (X+1)^2+(Y+1)^2-1 = 0$ and the line $Q_1(X, Y) = 1+X+Y = 0$ intersect the equilibrium points at $(-1, 0)$ and $(0, -1)$.

(ii) Invariant conditions: $A = 1/2$, $R = 1/2$, and $R' = 2$,

$$I_1(X, Y) = (X + X^2 + XY + CY^2)Y^{-2}$$

where $C$ is an arbitrary constant. Here a family of conic sections $P_1(X, Y) = X + X^2 + XY + CY^2 = 0$ intersect the line $Q_1(X, Y) = Y = 0$ at the equilibrium points $(0, 0)$ and $(-1, 0)$.

(iii) A symmetric case to (ii), with invariant conditions: $A = 2$, $R = 2$, and $R' = 1/2$,

$$I_1(X, Y) = (Y + XY + CX^2 + Y^2)X^{-2}$$

where $C$ is again an arbitrary constant. Here a family of conic sections $P_1(X, Y) = Y + XY + CX^2 + Y^2 = 0$ intersect the line $Q_1(X, Y) = x = 0$ at the equilibrium points $(0, 0)$ and $(0, -1)$.

All the invariant conditions here satisfy invariant condition (I) $RR' = 1$ in [2]. However, these more restrictive cases, with three invariant conditions instead of one, give time-independent invariant, whereas invariants of type (I) in [2] are all time dependent. In addition, invariant conditions here also satisfy invariant condition (III) $R + R'A = A + 1$ in [2], which also leads to a time-independent invariant. However, since the exponents of invariants of type (III) are proportional to $(RR' - 1)^{-1}$, it is clear that invariant conditions (I) and (III) are mutually exclusive (i.e. invariant conditions (I) and (III) cannot be satisfied simultaneously), a fact not explicitly stated in [2].

## 6. Summary and conclusions

From the phase portraits, we notice that if non-degenerate invariants of the forms $I_1$ and $I_2$ exist (i.e. the invariant conditions do not impose that the origin is the only equilibrium point; see case (iv), section 3), then for certain sets of coefficients of the QS the local phase-space in the neighbourhood of the equilibrium points is divided into a region bounded by polynomial curves (a line and a conic for $I_1$, three lines forming a triangle for $I_2$) and a region outside. For all non-degenerate $I_2$ invariants, the bounded region is a triangle with vertices at three equilibrium points (see figures 2 and 7) and trajectories starting inside will evolve to an attracting equilibrium point at one vertex of the triangle (since $\mu\nu < 0$). In contrast, a bounded region exists only for some $I_1$ invariants (e.g. see figures 1, 3 and 5). For these cases, trajectories starting inside the bounded region either remain inside ($\mu > 0$), where there is a centre and they are periodic and closed, or they approach an attracting equilibrium point ($\mu < 0$), indicating their asymptotic behaviour.

All the non-degenerate invariants found so far for the QS and LVS have three invariant conditions due to the difference between the number of equations and the number of unknowns. The Hamiltonian method yields ten equations corresponding to the coefficients of the $x$, $y$, $x^2$, $y^2$ and $xy$ terms in the $\dot{x}$ and $\dot{y}$ equations (see (8) and (10)). For the $I_1(x, y)$ invariant, there are actually four unknowns from the coefficients of the quadratic $P_1(x, y)$ polynomial and one unknown from the coefficients of the linear $Q_1(x, y)$ polynomial (the number of unknowns are less by one than the total number of coefficients of the polynomials, because a non-zero coefficient can always be factorized out and ignored: see discussions in case (i), section 3), and lastly, there is the unknown exponent $\mu$. For the $I_2(x, y)$ invariant, there are also only six actual unknowns: two from the coefficients of $P_2$ and one each from $Q_2$ and $R_2$, and the exponents $\mu$ and $\nu$. Thus both types of invariants considered here contain a total of six unknowns embedded in ten equations which *a priori* leaves four equations to impose conditions on the coefficients of the QS. However, the solutions sometimes eliminate one more equation than the number of unknowns suggests, so that only three equations are left to impose invariant conditions on the coefficients of the QS. For example, in section 3, case (iii), the solution $\mu = -1$ eliminates the two homogeneous equations in (8), and in section 4, the solution $\mu = -\nu$ also eliminates the two homogeneous solutions of (10). For other cases in section 3, it is not evident whether the solutions or the invariant conditions eliminate one more equation than their respective numbers, but nevertheless the unknown coefficients and the invariant conditions together eliminate one more equation than their total number. This reasoning remains valid for the LVS studied in section 5, where rescaling revealed that it has only three independent coefficients ($R$, $R'$ and $A$), and thus it is not surprising that the three invariant conditions specify all of them completely (i.e. in terms of real numbers).

In conclusion, we have found, along with results from [6], all the time-independent invariants of the forms $I_1 = P_1(x, y)Q_1(x, y)^{\mu}$ and $I_2 = P_2(x, y)Q_2(x, y)^{\mu}R_2(x, y)^{\nu}$

involving two and three equilibrium points of the QS, respectively. A natural extension based on geometrical suggestions from the phase portraits would be to find invariants that involve four equilibrium points of the QS, such as $I_3 = P_3(x, y)Q_3(x, y)^{\mu}R_3(x, y)^{\nu}S_3(x, y)^{\lambda}$ with $P_3$, $Q_3$, $R_3$ and $S_3$ linear polynomials in $(x, y)$, where two of the polynomials are homogeneous (i.e. vanish at the origin), to represent lines intersecting the origin. We expect to solve for ten unknown coefficients: four from the two non-homogeneous polynomials, two from the two homogeneous ones and one from the unknown product function $Q_3^{\mu-1}R_3^{\nu-1}S_3^{\lambda-1}F$ (substitute $I_3$ into (2) and factor out $Q_3^{\mu-1}R_3^{\nu-1}S_3^{\lambda-1}$) which has to be inversely proportional to a homogeneous polynomial, say $T(x, y)$, and the three numbers $\mu$, $\nu$ and $\lambda$. These unknowns are embedded in fourteen equations representing the coefficients of $x^2$, $y^2$, $xy$, $x^3$, $x^2y$, $xy^2$ and $y^3$ from $T(x, y)\dot{x}$ and $T(x, y)\dot{y}$ in (2). Hence, *a priori*, we expect the remaining four equations to impose perhaps four (or three) invariant conditions. Actually, solving the fourteen complicated coupled nonlinear equations is left for future work.

We hope that by fully exploring and utilizing the intimate relationship between the geometry of the equilibrium points and time-independent invariants for the two-dimensional QS, we might in the future apply these ideas to search for invariants in higher-dimensional autonomous systems. It is well known that higher-dimensional systems can exhibit chaotic orbits (e.g. the Lorenz system) and the existence of invariants, hence implying integrability of the system, will prevent trajectories from becoming chaotic [13]. Ever since the revival of the Painlevé method via the ARS algorithm [14], which gives conditions for integrability and thus the existence of invariants, there have been many works applying the ARS algorithm to three-dimensional systems, such as the reduced LVS known as the ABC system (the most recent and complete work being [15]) and the Lorenz system [13]. A resulting important conjecture by Tabor and Weiss for the Lorenz system [13] is whether satisfying just partial Painlevé conditions is sufficient for the existence of invariants. Unfortunately, the Carleman method seems to have yielded all the known (six) time-dependent invariants for the Lorenz system in which one set of partial Painlevé conditions is sufficient to provide an invariant while the other set is not sufficient [9, 13]. Presently, the issue of the sufficiency of the partial Painlevé conditions is far from settled. The Hamiltonian method, with the help of geometrical ideas, has found other invariants which the Carleman method left out (as was shown in section 5 for the two-dimensional LVS), and we hope in future, using the Hamiltonian method, to be able to find other invariants for the Lorenz system which might settle the conjecture of the partial Painlevé conditions for integrability.

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## Appendix

For the case in which three lines intersect the origin, we choose for $I_2 = P_2\ Q_2^{\mu}\ R_2^{\nu}$

$$
\begin{aligned}
P_2(x, y) &= \alpha x + \beta y \\
Q_2(x, y) &= Ax + By \\
R_2(x, y) &= Cx + Dy.
\end{aligned}
$$

As in sections 2 and 4, we substitute the expressions for $P_2$, $Q_2$ and $R_2$ into (6) and equate the appropriate coefficients to those of the QS (after setting $Q_2^{\mu-1}R_2^{\nu-1}F = 1$). As in similar cases, the solutions to the coefficients and exponents are too algebraically involved, so that we decided to present them in a simplified manner as

$$\alpha = \frac{-DBc_2}{c_1} \qquad \beta = 1$$
$$A = 1 \qquad C = 1$$
$$\mu = \frac{\left(c_1 + c_2D^2B\right)\left(c_2D^2B + b_{11}DB - b_{22}D - c_1\right)}{\left(c_1c_2D + b_{22}c_2DB + c_1b_{11} + c_1c_2B\right)(D - B)D}$$
$$\nu = -\frac{\left(c_1 + c_2DB^2\right)\left(c_2DB^2 + b_{11}DB - b_{22}B - c_1\right)}{\left(c_1c_2D + b_{22}c_2DB + c_1b_{11} + c_1c_2B\right)(D - B)B}$$

with the coefficients $B$ and $D$ being solutions to the two algebraic equations,

$$c_2D^2B^2 + (b_{12} - b_{22})DB - c_1(D + B) = 0$$
$$c_2D^2B + (b_{11} - b_{21})DB + c_2DB^2 - c_1 = 0.$$

There are only two invariant conditions, $a_1 = 0 = a_2$, which makes this case more general than case (iv) in section 3, which also has these two invariant conditions in addition to a third. In fact, after some cumbersome numerical work, we could show that imposing the third invariant condition from case (iv) of section 3 to this case here, we obtain the expected result that $\mu = \nu$. As in case (iv) of section 3, this degenerate case has only one equilibrium point at the origin.

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